A simple iterative algorithm for solving assignment problems

Amir Egozi and Yosi Keller and Hugo Guterman

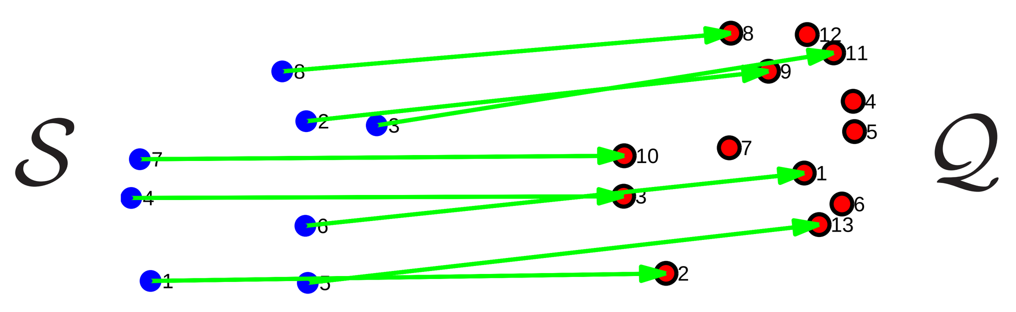
Department of Electrical and Computer Engineering, Ben-Gurion University of the Negev, Israel

Introduction

Finding a mapping between one set of points and another set of points is the bottle neck of many computer vision applications, e.g., multiple view reconstruction, image registration and model-based object recognition.

Setup:

- $\mathcal{S} = \{s_i\}_{i=1}^{n}$ and $\mathcal{Q} = \{q_i\}_{i=1}^{m}$ represent two point sets in $\mathbb{R}^d$ ($|\mathcal{S}| = n$, and $|\mathcal{Q}| = m$).
- $\mathcal{Q}$ contains a noisy replica of $\mathcal{S}$ undergoing some rigid transformation with a certain number of additional points, hence $n \leq m$.



Goal:

Find the assignment function,

- $f : \mathcal{S} \to \mathcal{Q}$, such that $f(s_i) \neq f(s_j)$ for all $s_i \neq s_j$ in $\mathcal{S}$,

that reveal the correct instance of $\mathcal{S}$ in $\mathcal{Q}$.

Definition

The *matching matrix* $X = (x_{ij}) \in \{0,1\}^{n \times m}$

$$x_{ij} = \begin{cases} 1 & \text{if } s_i \in \mathcal{S} \text{ match } q_j \in \mathcal{Q}, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\sum_{i=1}^{n} x_{ij} \leq 1, \quad \sum_{j=1}^{m} x_{ij} = 1.$$

The *matching vector* - $\mathbf{x} = \text{vect}(X)$ is a row-wise vectorization.

The Quadratic Assignment Problem (QAP)

- We are given a pairwise affinity measure (or compatibility),

$$\Omega : \mathcal{S}^2 \times \mathcal{Q}^2 \to \mathbb{R}^+,$$

that measure the cost of a pair of individual assignments.

- The Gaussian kernel is a common choice:

$$\Omega(s_i, s_j, q_{i'}, q_{j'}) = \exp\left\{ -\frac{(||s_i - s_j||_2 - ||q_{i'} - q_{j'}||_2)^2}{\sigma} \right\},$$

where $\sigma > 0$ is the kernel width.

- The L_2 norm can be replaced by any other intrinsic distance measure.
- The affinity matrix is a square symmetric matrix that consists of all pairwise affinities, $A = (a_{ij}) \in \mathbb{R}^{N \times nN}$,

$$a_{ij} = \Omega_2(s_i, s_j, q_{i'}, q_{j'})$$

The Quadratic Assignment Problem (QAP)

find the assignment function which has the maximum total affinity

$$\hat{f} = \arg\max_{f} \sum_{s_i \in \mathcal{S}} \sum_{s_j \in \mathcal{S}} \Omega(s_i, s_j, f(s_i), f(s_j))$$

this can be expressed as:

$$\hat{\mathbf{x}} = \arg\max_{\mathbf{x}} \mathbf{x}^T A \mathbf{x}$$

$$\text{s.t. } X\mathbf{1} = \mathbf{1}, \; X^T\mathbf{1} \leq \mathbf{1} \text{ and } \mathbf{x} \in \{0,1\}^N.$$

This optimization problem is NP-complete! [Shi and Malik TPAMI'00]

Probability interpretation

- Let $e = (s, q), s \in \mathcal{S}, q \in \mathcal{Q}$ denote an individual assignment, and the set of all potential assignments is:

$$E \subseteq \mathcal{S} \times \mathcal{Q},$$

where $|E| = N$.

- Each entry in $\mathbf{x}$ represent a potential mapping that is either chosen to be in the assignment function f or not.
- The confidence (probability) of $e_i, i = 1, \ldots, N$, to belong to the matching set is denoted by - $p(e_i \in M)$.
- These probabilities form a real vector $\mathbf{z}$, where

$$z_i = p(e_i) \in [0, 1].$$

Probabilities Updating Scheme

Consider the affinities a_{ij} as conditional probabilities,

$$a_{ij} = P(e_i \mid e_j).$$

Goal - to extract the marginal probabilities for each edge:

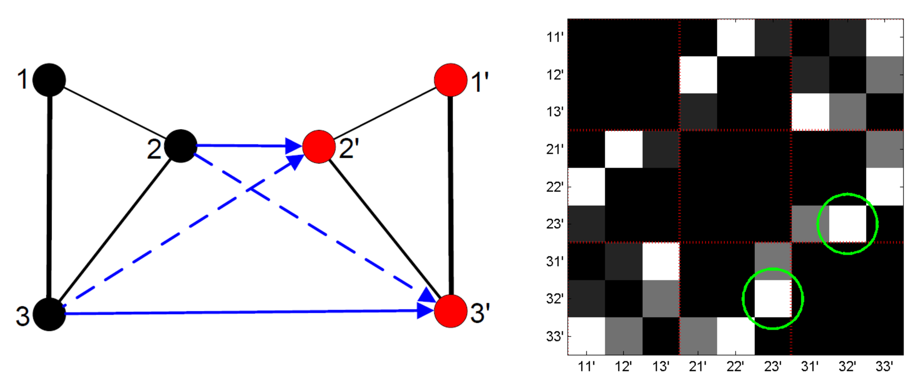
$$P(e_i) = \sum_j P(e_i, e_j) = \sum_j P(e_i \mid e_j) P(e_j) \qquad (1)$$

Problem - $P(e_i)$ is unknown.

Solution - initialize $P(e_i) = \frac{1}{m}$, and iterate the marginal formula.

Properties of Updating Scheme

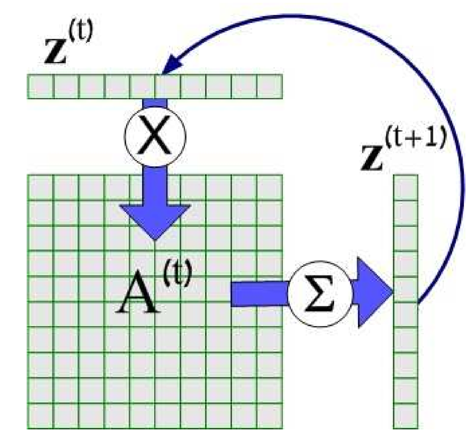
The updating scheme reduce the affect of high valued affinities (green circle in the right image) that are not consistent with the correct matching.



Iterative scheme:

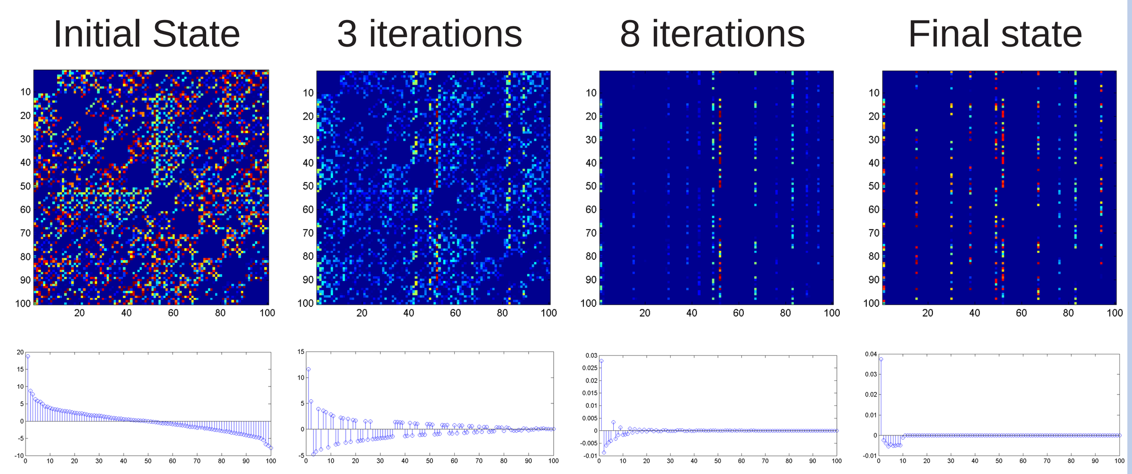
- One iteration of Eq. (1) consists of multiplying the columns of A by the corresponding entry in $\mathbf{z}$ and sum over the rows.
- The Affinity Matrix Evolution (AME) algorithm can be summarizes as:

$$\mathbf{z}^{(t+1)} = A^{(t)}\mathbf{z}^{(t)}, \quad A^{(t+1)} = A^{(t)}D^{(t)},$$



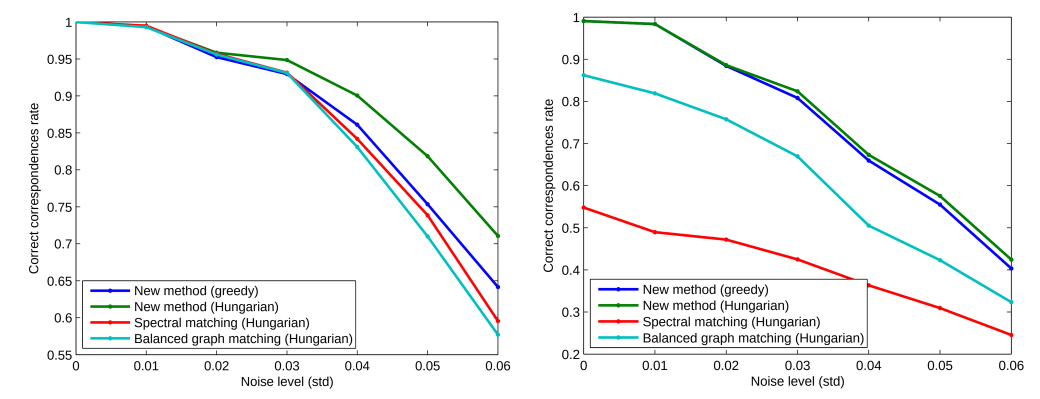
$t = 1, 2, \ldots,$
where $D^{(t)} = diag(z_1^{(t)}, \ldots, z_m^{(t)})$

Affinity Matrix Evolution and Its Spectrum



Experimental Results

Synthetic data: 20 points randomly generated in $[0,1] \times [0,1]$, undergo *nearly*-rigid transformation with increasing level of noise and 10 outliers (right), affinity parameter - $\sigma = 0.03$.



20×20 $\qquad$ 20×30

Conclusion

- We present a novel approach for approximating the mapping between two point sets.
- The approach is based on iteratively updating the affinities between pairwise individual assignments.
- As far as we know, our scheme is the first to iteratively modify the affinity matrix in order to obtain an approximate solution to the QAP.
- The proposed scheme can readily adopted for solving Multi-indexing Assignment Problems.