Spectral Symmetry Analysis

Michael Chertok and Yosi Keller

School of Engineering, Bar-Ilan University, Israel.
Accepted for publication to PAMI.

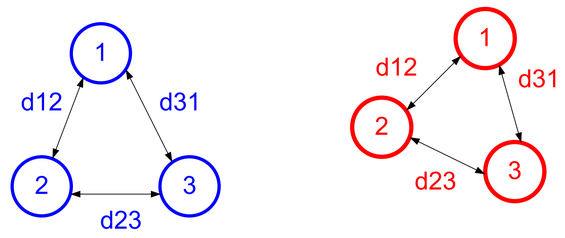
Introduction

We present a spectral approach for detecting and analyzing rotational and reflectional symmetries in n-dimensions.

- The problem is formulated as a self-alignment of a set of points
- Each object is represented by a set of points $S \in \mathbb{R}^n$
- The self-alignment problem is formulated as a quadratic binary optimization problem
- The optimization problem is solved by spectral relaxation
- Symmetric objects have more than one self alignments, resulting in
- A multiplicity of eigenvalues whose corresponding eigenvectors hide symmetric self-alignments
- Geometrical constrains further improve scheme's robustness

Matching of sets and subgraphs

Consider the following alignment problem



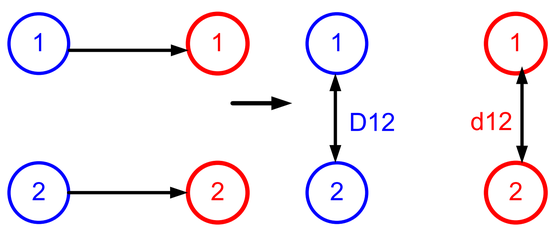
We are given two sets of points: S_k $k = 1, 2$ and the relative distances within each set: $d^k_{i,j}$ $i, j = 1..|S_k|$.

Definitions

Assignment vector $x = \left[0\ 1\ 0|\ 1\ 0\ 0|\ 0\ 0\ 1\right]^T$

Assignment of a point $S^i_1 \in S_1$ to a point $S^{i'}_2 \in S_2$: $C_{ii'} \equiv S^i_1 \rightarrow S^{i'}_2$

Assignment cost for pairs $d\left(C_{ii'}, C_{jj'}\right) = d\left(d_{ij}, d_{i'j'}\right) = \left|d_{ij} - d_{i'j'}\right|$



Spectral Graph Matching

Assignment affinity matrix $a_{ii',jj'} = \exp\left(-d\left(d_{ij}, d_{i'j'}\right)/\sigma\right)$, $\sigma > 0$
We can now define the total assignment affinity and maximize it:

$$\mathbf{x}^* = \arg\max_{\mathbf{x}} \mathbf{x}^T A \mathbf{x} = \arg\max_{\mathbf{x}} \sum_{i,j \in x} A_{ij}$$

It is easy to show that A is p.s.d. This is a difficult optimization problem:
So **Relax x**

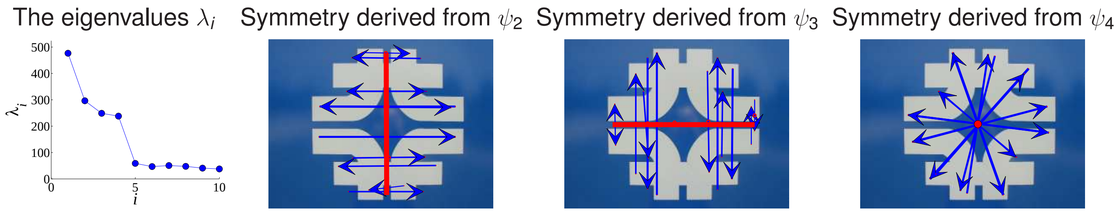
$$\mathbf{x}^* = \arg\max_{\mathbf{x}} \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}},\ \mathbf{x} \in \mathbb{R}$$

- $\mathbf{x}^*$ is the eigenvector ψ_1 corresponding to the largest eigenvalue λ_1
- $\mathbf{x}^*$ is discretized into binary assignment vector $\mathbf{x}_d \in \{0, 1\}$.

Spectral Matching for Self-Alignments

- Align the set of points to itself : $S_1 \equiv S_2$
- Alignment derived from ψ_1 is trivial (each point to itself)
- For symmetric objects ψ_i, $i = 2..K$, hide symmetric self-alignments C_i
- The eigengap in the graph of eigenvalue signals the symmetry order

The eigenvalues λ_i | Symmetry derived from ψ_2 | Symmetry derived from ψ_3 | Symmetry derived from ψ_4



Symmetry classification and pruning false symmetries

Geometric verification is used to classify the detected symmetry as reflectional or rotational symmetry and to prune false detections.

$$T_{rot}(x, y) = \begin{pmatrix} \cos\beta_k & -\sin\beta_k & 0 \\ \sin\beta_k & \cos\beta_k & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \quad (1)$$

The center of rotation X_c, is invariant under T_{rot}, and can be computed as the eigenvector of T_{tor} corresponding to the eigenvalue $\lambda = 1$

$$T_{rot} X_c = X_c. \quad (2)$$

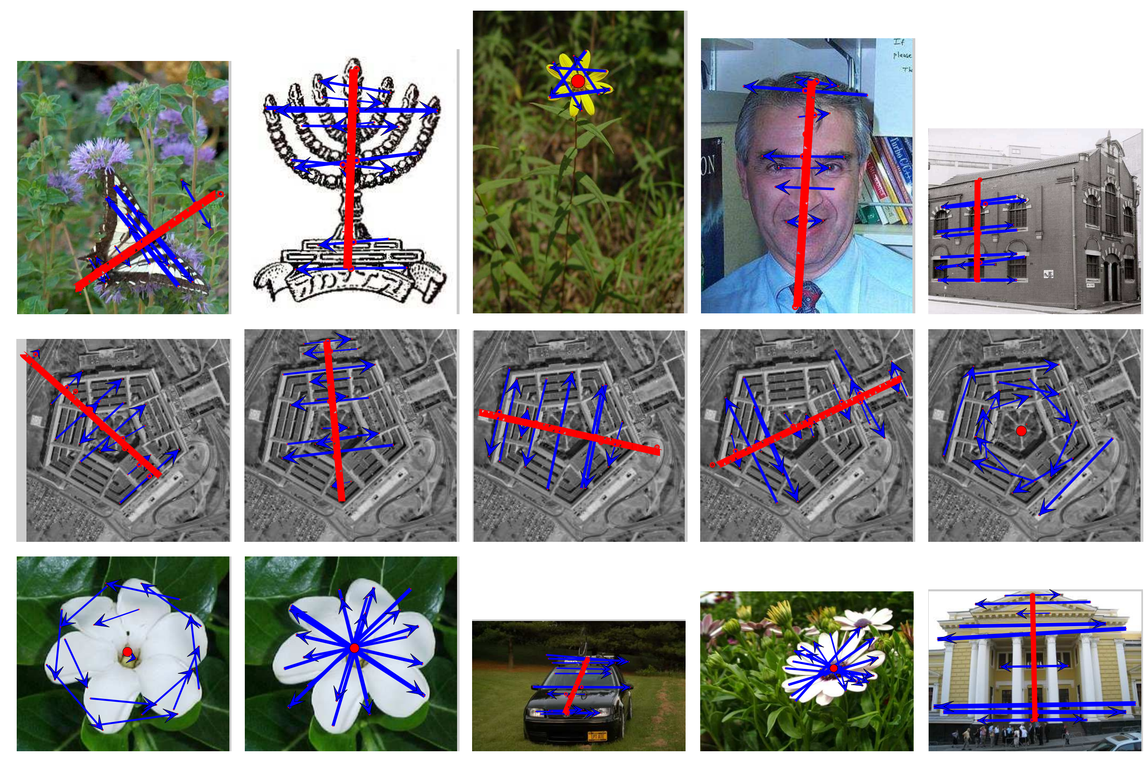
$$T_{ref}(x, y) = \begin{pmatrix} \cos 2\alpha_0 & \sin 2\alpha_0 & 0 \\ \sin 2\alpha_0 & -\cos 2\alpha_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \quad (3)$$

The points on the symmetry axis form an invariant set X_R that corresponds to the eigenspace of

$$T_{ref} X_R = X_R. \quad (4)$$

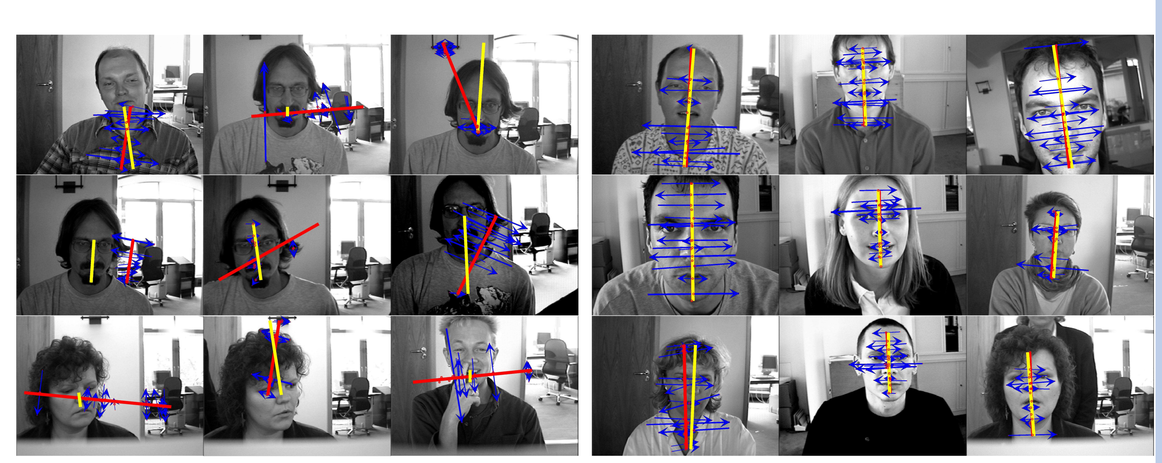
- Derive correspondence map C_i from ψ_i
- Estimate transformation matrix T_i
- If $|\det(T_i) - 1| < \varepsilon \Rightarrow$ rotation detected, use (2) to find symmetry center
- If $|\det(T_i) + 1| < \varepsilon \Rightarrow$ reflection detected, use (4) to find symmetry axis
- Otherwise, classify this C_i as false detection

Experimental Results



BioDB Benchmark

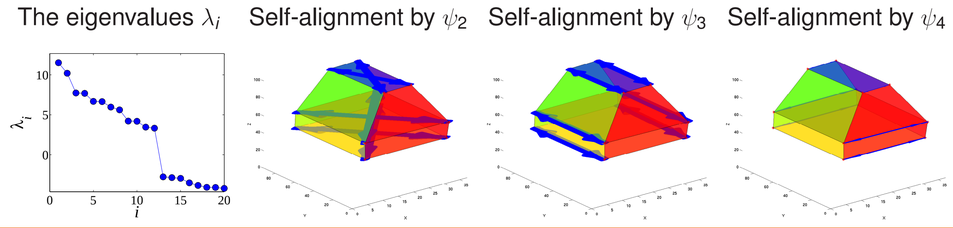
- BioID dataset consists of 1521 facial images with ground truth
- We detected symmetry in faces with 99.41% accuracy
- The only 9 miss detections are on the left. Examples of correct detections are on the right :



Three-dimensional symmetry

Spectral Symmetry Analysis can be applied naturally to $\mathbb{R}^3$
Pairwise distances $d_{ii'}$ are calculated in $\mathbb{R}^3$

The eigenvalues λ_i | Self-alignment by ψ_2 | Self-alignment by ψ_3 | Self-alignment by ψ_4



Mail: yosi.keller@gmail.com

WWW: http://www.eng.biu.ac.il/~kellery1/results/face-analysis.html